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Geometric Kac–Moody modularity

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Abstract

It is shown how the arithmetic structure of algebraic curves encoded in the Hasse–Weil L-function can be related to affine Kac–Moody algebras. This result is useful in relating the arithmetic geometry of Calabi–Yau varieties to the underlying exactly solvable theory. In the case of the genus three Fermat curve we identify the Hasse–Weil L-function with the Mellin transform of the twist of a number theoretic modular form derived from the string function of a non-twisted affine Lie algebra. The twist character is associated to the number field of quantum dimensions of the conformal field theory.

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1. Introduction

1.1 Number theoretic methods have proven useful in attempts to understand string theoretic aspects of Calabi–Yau varieties. Physically, string theory is a two-dimensional conformal field theory on a Riemannian surface. Mathematically it can be viewed, in the present context, as a vertex operator algebra associated to an affine Kac–Moody algebra. The problem of string compactification can be interpreted as an attempt to construct a map which relates Kac–Moody vertex algebras to the geometry of Calabi–Yau varieties. Part of the data of the conformal field theory are the anomalous dimensions of the fields of the theory. These scaling dimensions are rational numbers that appear in the correlation functions of the quantum field theory. It turns out that it is more useful to think about these values in terms of the number field generated by their associated quantum dimensions, defined via the characters of the Kac–Moody algebra. A relation between these quantum dimensions and the arithmetic of Calabi–Yau varieties of Brieskorn–Pham type has been established in [1] by considering the Hecke theoretic nature of the Hasse–Weil L-functions of these varieties. A more conceptual framework of this relation has been formulated in [2].

An important aspect of conformal field theory is modular invariance, a property of the string that appears difficult to explain from a geometric point of view. It has in particular been an open question for a long time how the string theoretic building blocks on the world sheet are reflected in the geometry of spacetime. One way to make this question more precise is by asking how the characters that appear in the partition function of the string emerge from spacetime, and how in turn the spacetime geometry can be constructed from string theoretic quantities. Questions of this type have a long history in arithmetic algebraic geometry, e.g. in the context of the Shimura–Taniyama conjecture [3,4] and the Langlands' program, but string theory provides a somewhat different focus than the one encountered in a broader arithmetic framework. Modularity of motivic L-functions associated to algebraic varieties along the lines of Langlands' program is not sufficient. In order to be of string theoretic significance, the motivic modular forms should allow an interpretation in terms of modular forms that arise in the description of the physics on the Riemann surfaces defined by the propagating string.

It was shown in [5] in the context of string compactifications on elliptic curves that the Mellin transform of the Hasse–Weil L-function of the plane cubic

$$C_3 = \{(x : y : z) \in \mathbb{P}_2 | x^3 + y^3 + z^3 = 0\} \tag{1}$$

factors into a product of modular forms that arise from the characters of the underlying two-dimensional theory. More precisely, the following result was obtained. Define $q = \mathrm{e}^{2\pi i\tau}$ and let

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n) \tag{2}$$

denote the Dedekind η-function, $c^k_{\ell,m}(\tau)$ be the affine SU(2) string functions at conformal level $k \in \mathbb{N}$, and

$$\Theta^k_{\ell,m}(\tau) = \eta^3(\tau)c^k_{\ell,m}(\tau) \tag{3}$$

be the Hecke indefinite modular forms associated to $c^k_{\ell,m}(\tau)$. Define further the congruence group of elements of $\mathrm{SL}(2,\mathbb{Z})$ that are upper triangular mod N as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\, N) \right\}, \tag{4}$$

and denote by $S_2(\Gamma_0(N))$ the space of cusp forms with respect to $\Gamma_0(N)$.

Theorem 1.1. *The Mellin transform of the Hasse–Weil L-function $L_{\mathrm{HW}}(C_3, s)$ of the cubic elliptic curve $C_3 \subset \mathbb{P}_2$ is a modular form $f_{\mathrm{HW}}(C_3, q) \in S_2(\Gamma_0(27))$ which factors into the product*

$$f_{\mathrm{HW}}(C_3, q) = \Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9). \tag{5}$$

Here $\Theta^1_{1,1}(\tau) = \eta^3(\tau)c^1_{1,1}(\tau)$ is the Hecke modular form associated to the quadratic extension $\mathbb{Q}(\sqrt{3})$ of the rational field $\mathbb{Q}$, determined by the unique string function $c^1_{1,1}(\tau)$ of the affine Kac–Moody SU(2)*-algebra at conformal level $k = 1$.*

1.2 Elliptic curves are somewhat degenerate examples of Calabi–Yau manifolds and the question arises whether the results of [5] can be extended to more general algebraic curves, e.g.

$$C_n = \{(x : y : z) \in \mathbb{P}_2 | x^n + y^n + z^n = 0\}. \tag{6}$$

The genus of these Riemann surfaces is given by $g(C_n) = (n-1)(n-2)/2$, hence except for the cubic curve C_3 just discussed these curves are not elliptic. It is therefore not obvious whether one should be able to find a string theoretic interpretation, generalizing the result described above for C_3. For such general Riemann surfaces there exists in particular no a priori statement like the proof of the Shimura–Taniyama conjecture which would ensure the modularity of the q-expansion derived from the Hasse–Weil L-function. It is this question which we address in this paper.

An argument in favor of a conformal field theoretic interpretation of the curves C_n is provided by the fact that they appear as singular sets of higher dimensional Calabi–Yau varieties that are expected to be exactly solvable. For the cubic curve C_3 such higher dimensional manifolds are abundant, examples being provided by the elliptic K3 surface defined by the degree 6 polynomial in weighted projective space $\mathbb{P}_{(1,1,2,2)}$

$$S_6 = \{(z_1 : \cdots : z_4) \in \mathbb{P}_{(1,1,2,2)} | z_1^6 + z_2^6 + z_3^3 + z_4^3 = 0\}, \tag{7}$$

or the degree 12 Calabi–Yau threefold

$$X_{12} = \{(z_0 : \cdots : z_4) \in \mathbb{P}_{(1,1,2,4,4)} | z_0^{12} + z_1^{12} + z_2^6 + z_3^3 + z_4^3 = 0\}. \tag{8}$$

To be more precise, the Gepner model of X_{12} is given by

$$\mathbb{P}_{(1,1,2,4,4)} \supset X_{12} \cong (10_A^2 \otimes 4_A \otimes 1_A^2)_{\mathrm{GSO}}, \tag{9}$$

with two minimal models at conformal level $k = 10$, one model at $k = 4$, and two models at $k = 1$, all equipped with the diagonal affine invariant. The singular curve in this threefold is $\mathbb{P}_{(1,2,2)}[6] \cong \mathbb{P}_2[3]$, i.e. the Fermat curve C_3. The result of Theorem 1.1 provides a geometric construction of the characters of the two minimal factors at $k = 1$.

In the case of X_{12} we can also identify the curve C_3 as the generic elliptic fiber which can be identified by an iterative application of the construction described in [6]. First we can construct the elliptic K3 surface S_6 via the twist map as

$$\mathbb{P}_{(2,1,1)}[6] \times \mathbb{P}_2[3] \to \mathbb{P}_{(1,1,2,2)}[6], \tag{10}$$

and then the hypersurface X_{12} can be constructed by applying the twist map again in the form

$$\mathbb{P}_{(2,1,1)}[12] \times \mathbb{P}_{(1,1,2,2)}[6] \to \mathbb{P}_{(1,1,2,4,4)}[12]. \tag{11}$$

We therefore see that the twist map allows to construct the threefold X_{12} as an iterated orbifolds of products of curves and makes explicit the origin of the two minimal factors at $k = 1$. There is a large number of varieties of this type among the Calabi–Yau hypersurface threefolds constructed in the mirror paper [7], and in refs. [8], which describe the complete construction of this class of varieties.

For higher genus curves of type C_n similar embeddings can be obtained. A few examples are provided by the quartic plane curve C_4, which is embedded in the octic hypersurface in weighted projective space $\mathbb{P}_{(1,1,2,2,2)}$, the quintic curve C_5, embedded in $\mathbb{P}_{(1,3,2,2,2)}$, or the septic curve C_7 in $\mathbb{P}_{(1,7,2,2,2)}$. The class of higher dimensional varieties of this type is large. Because they form part of an exactly solvable variety we might expect these curves to inherit the information of the underlying string theory. In this way we are led to ask whether more general, non-Calabi–Yau varieties are related to affine Kac–Moody algebras similar to the cubic.

In the present paper we first analyze the simplest non-elliptic Fermat curve C_4 in detail, and in the last section indicate a procedure which puts the quartic into perspective and shows that other curves can be treated in a similar way. In particular we prove the following result.

Theorem 1.2. *The Hasse–Weil L-function $L_{\mathrm{HW}}(C_4, s)$ of the quartic plane curve C_4 factors into a triple product $L_{\mathrm{HW}}(C_4, s) = L_{\mathrm{HW}}(E_4, s)^3$, where $L_{\mathrm{HW}}(E_4, s)$ is Hasse–Weil L-functions of the weighted elliptic curve*

$$E_4 = \{(x_0 : x_1 : x_2) \in \mathbb{P}_{(1,1,2)} | x_0^4 + x_1^4 + x_2^2 = 0\}. \tag{12}$$

The Hasse–Weil modular form $f_{\mathrm{HW}}(E_4, q) \in S_2(\Gamma_0(64))$ associated to $L(E_4, s)$ factors into a twisted product

$$f_{\mathrm{HW}}(E_4, q) = \Theta^2_{1,1}(4\tau)^2 \otimes \chi_2. \tag{13}$$

Here the twist character is the Legendre symbol $\chi_2(\cdot) = (\frac{2}{\cdot})$ and $\Theta^2_{1,1}(\tau)$ is the affine $SU(2)$ theta function of the string function $c^2_{1,1}(\tau)$ at conformal level $k = 2$.

This result shows that the modular forms derived from the quartic curve admits a Kac–Moody theoretic interpretation via an SU(2) theta function.

1.3 We can turn this result around and read it as an arithmetic interpretation of a conformal field theoretic object, the string theoretic parafermionic string functions.

Corollary 1.3. *The* SU(2) *string function* $c^2_{1,1}(\tau)$ *at level* $k = 2$ *is determined by the twisted Hasse–Weil modular form*

$$c^2_{1,1}(\tau) = \frac{1}{\eta^3(\tau)}\sqrt{f_{\mathrm{HW}}(E_4) \otimes \chi_2(q^{1/4})}. \tag{14}$$

1.4 The weighted elliptic curve E_4 which emerges as the arithmetic building block of the quartic curve also appears as the singular set of many higher dimensional varieties and it is possible to repeat the conformal field theoretic analysis indicated above for Fermat curves. This can be illustrated by considering the Calabi–Yau threefold X_{16} of degree 16 embedded in the weighted projective space $\mathbb{P}_{(1,1,2,4,8)}$. This variety is a K3-fibration with a generic fiber described by a degree 8 hypersurface S_8 embedded in $\mathbb{P}_{(1,1,2,4)}$ which is elliptic. The generic fiber of this elliptic fibration is precisely the curve E_4 and we can iteratively construct the degree 16 threefold in $\mathbb{P}_{(1,1,2,4,8)}$ with the methods of [6] by first considering the map

$$\mathbb{P}_{(2,1,1)}[8] \times \mathbb{P}_{(1,1,2)}[4] \to \mathbb{P}_{(1,1,2,4)}[8], \tag{15}$$

and then applying the map

$$\mathbb{P}_{(2,1,1)}[16] \times \mathbb{P}_{(1,1,2,4)}[8] \to \mathbb{P}_{(1,1,2,4,8)}[16]. \tag{16}$$

Alternatively, E_4 appears as the $\mathbb{Z}_2$-singular curve on this threefold. As in the case of Theorem 1.1 in the case of the cubic curve Theorem 1.2 provides a geometric interpretation of the string theoretic modular form of the minimal factor at conformal level $k = 2$ which is part of the Gepner model corresponding to this threefold.

It becomes clear by an iterative application of the twist map construction [6] that elliptic threefolds which are K3 fibrations can be built from the generic elliptic fiber and other curves. Thereby the question of modularity for these higher dimensional varieties is reduced to the modularity problem of curves.

1.5 This paper is organized as follows. In Section 2 we compute the Hasse–Weil L-function of the quartic. We will see that arithmetically the basic building block of this curve is given by the elliptic Brieskorn–Pham curve of degree 4. In Section 3 we briefly review the relevant aspects of non-twisted affine Kac–Moody algebras. In Section 4 the modular form defined by the twisted Mellin transform of the Hasse–Weil L-function is related to a modular form derived from the character of an affine Kac–Moody algebra. A relation between the character defining this twist and the quantum dimensions of the affine Kac–Moody algebra is described in Section 5. In the final Section 6 we show that the factorization behavior of the quartic is typical for Fermat curves, indicating that other curves can be treated in a similar way.

2. Hasse–Weil L-function

2.1 For algebraic varieties X the congruent zeta function at a prime number p can be defined as the generating function

$$Z(X/\mathbb{F}_p, t) \equiv \exp\left(\sum_{r\in\mathbb{N}} \#(X/\mathbb{F}_{p^r})\frac{t^r}{r}\right). \tag{17}$$

It was first shown by Schmidt [9,10] that for algebraic curves X the zeta function $Z(X/\mathbb{F}_p, t)$ is a rational function which takes the form

$$Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}^{(p)}(t)}{(1-t)(1-pt)}, \tag{18}$$

where

$$\mathcal{P}^{(p)}(t) = \sum_{i=0}^{2g} \beta_i(p) t^i, \tag{19}$$

is a polynomial whose degree is given by the genus $g(X) = (2-\chi(X))/2$ of the curve, where $\chi(X)$ is the Euler characteristic of the curve. More important is the global zeta function obtained by setting $t = p^{-s}$ and taking the product over all rational primes at which the variety has good reduction. Let S denote the set of rational primes at which X becomes singular and denote by P_S the set of primes that are not in S. The global zeta function then becomes

$$Z(X, s) = \prod_{p\in P_S} \frac{\mathcal{P}^{(p)}(p^{-s})}{(1-p^{-s})(1-p^{1-s})} = \frac{\zeta(s)\zeta(s-1)}{L_{\mathrm{HW}}(X, s)}, \tag{20}$$

with the Hasse–Weil L-function

$$L_{\mathrm{HW}}(X, s) = \prod_{p\in P_S} \frac{1}{\mathcal{P}^{(p)}(p^{-s})}, \tag{21}$$

and the Riemann zeta function $\zeta(s) = \prod_p (1-p^{-s})^{-1}$.

The most direct way to compute the expansion at least for the lower primes involves the comparison of the definition of the congruent zeta function with the rational expression found by Schmidt. For our purposes it suffices to collect the first three non-trivial coefficients

$$\begin{aligned}
\beta_0(p) &= 1,\\
\beta_1(p) &= N_{1,p} - (p+1),\\
\beta_2(p) &= \tfrac{1}{2}(N_{1,p}^2 + N_{2,p}) - (p+1)N_{1,p} + p,\\
\beta_3(p) &= \tfrac{1}{3}N_{3,p} + \tfrac{1}{2}N_{2,p} + \tfrac{1}{6}N_{1,p}^3 - \tfrac{1}{2}(N_{1,p}^2 + N_{2,p}) + pN_{1,p},\\
&\vdots\\
\beta_6(p) &= p^3.
\end{aligned} \tag{22}$$

We collect in Table 1 our results for the first few primes.

Table 1
The coefficients β_1, β_2 for the quartic curve C_4 of genus 3

	Prime p												
	2	3	5	7	11	13	17	19	23	29	31	37	41
$N_{1,p}$	3	4	0	8	12	32	12	20	24	0	31	32	12
$N_{2,p}$	5	28	34										
$\beta_1(p)$	0	0	−6	0	0	18	−6	0	0	−30	0	−6	−30
$\beta_2(p)$	0	9	27										

Using these results leads to the expansion

$$L_{\rm HW}(C_4, s) = \prod_p \frac{1}{1 + \beta_1(p)p^{-s} + \cdots + p^3 p^{-6s}}$$
$$= 1 + \frac{6}{5^s} - \frac{9}{9^s} - \frac{18}{13^s} + \frac{6}{17^s} + \frac{9}{25^s} + \frac{30}{29^s} + \frac{6}{37^s} + \frac{30}{41^s} + \cdots \tag{23}$$

The associated q-series then takes the form

$$f_{\rm HW}(C_4, q) = q + 6q^5 - 9q^9 - 18q^{13} + 6q^{17} + 9q^{25} + 30q^{29} + 6q^{37} + 30q^{41} + \cdots \tag{24}$$

Finite expansions like this often turn out to be useful because of theorems by Faltings and Serre which show that such functions are determined uniquely by a finite number of terms.
2.2 The behavior of the coefficients a_n of the Hasse–Weil q-expansion under Hecke operators indicates that this series does not describe a Hecke eigenform. Hence the above result for the Hasse–Weil L-function is not particularly illuminating except for the fact that all the coefficients are divisible by 3. This suggests that the Hasse–Weil L-function of the quartic plane curve can be viewed as the cubic power of a more basic L-series. We can write $L_{\rm HW}(C_4, s) = L^3(s)$ with

$$L(s) = 1 + \frac{2}{5^s} - \frac{3}{9^s} - \frac{6}{13^s} + \frac{2}{17^s} - \frac{1}{25^s} + \frac{10}{29^s} + \frac{2}{37^s} + \frac{10}{41^s} + \cdots \tag{25}$$

The series $L(s)$ can be obtained via the Mellin transform

$$L(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(iy) y^{s-1}\, {\rm d}y \tag{26}$$

from the expansion

$$f(q) = q + 2q^5 - 3q^9 - 6q^{13} + 2q^{17} - q^{25} + 10q^{29} + 2q^{37} + 10q^{41} + \cdots \tag{27}$$

This result indicates that $f(q)$ is the basic building block of the Hasse–Weil L-form $f_{\rm HW}(C_4, q)$, providing a sort of cubic root of it via its L-series.
2.3 The question arises whether one can interpret the expansion (27) in a geometric way as the Hasse–Weil modular form of some other geometric object. To answer such a factorization question it is useful to understand the polynomials $\mathcal{P}^{(p)}(t)$ which determine the congruence zeta function in a more systematic way in terms of Jacobi sums.

Theorem ([11]). *For the plane curve*

$$C_n = \{z_0^n + z_1^n + z_2^n = 0\} \subset \mathbb{P}_2 \tag{28}$$

defined over the finite field $\mathbb{F}_q$ *set* $d = (n, q-1)$ *and define the set* $\mathcal{A}_2^{q,n}$ *of triplets* $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ *of rational numbers*

$$\mathcal{A}_2^{q,n} = \left\{(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Q}^3 | 0 < \alpha_i < 1, d = (n, q-1), \mathrm{d}\alpha_i \right.$$
$$\left. = 0\,(\mathrm{mod}\ 1),\ \sum_{i=0}^{2} \alpha_i = 0\,(\mathrm{mod}\ 1)\right\}.$$

For such triplets define the Jacobi sums

$$j_q(\alpha_0, \alpha_1, \alpha_2) = \frac{1}{q-1} \sum_{\substack{u_0, u_1, u_2 \in \mathbb{F}_q \\ u_0 + u_1 + u_2 = 0}} \chi_{\alpha_0}(u_0)\chi_{\alpha_1}(u_1)\chi_{\alpha_2}(u_2), \tag{29}$$

where $\chi_{\alpha_i}(u_i) = \mathrm{e}^{2\pi i \alpha_i m_i}$ *and the integers* m_i *are determined via* $u_i = g^{m_i}$, *where* $g \in \mathbb{F}_q$ *is a generator. Then the cardinality of* $C_n/\mathbb{F}_q$ *is given by*

$$\#(C_n/\mathbb{F}_q) = N_{1,q}(C_n) = 1 + q + \sum_{\alpha \in \mathcal{A}_2^{q,n}} j_q(\alpha_0, \alpha_1, \alpha_2). \tag{30}$$

With these Jacobi sums one can express the F.K. Schmidt polynomial as

$$\mathcal{P}^{(p)}(t) = \prod_{\alpha \in \mathcal{A}_2^n} (1 + j_{p^{\mu_\alpha}}(\alpha) t^{\mu_\alpha})^{1/\mu_\alpha}, \tag{31}$$

where μ_α is the smallest positive integer such that for α in the set $\mathcal{A}_2^n$ defined by

$$\mathcal{A}_2^n = \left\{\alpha \in \mathbb{Q}^3 | \alpha_i \in (0,1), n\alpha_i \in \mathbb{N}, \sum_{i=0}^{2} \alpha_i \in \mathbb{N}\right\}, \tag{32}$$

one finds $(p^{\mu_\alpha} - 1)\alpha_i \in \mathbb{N}$ for all i. Applied to the quartic curve C_4 the set $\mathcal{A}_2^{q,4}$ is given by

$$\mathcal{A}_2^{q,4} = \begin{cases} \emptyset, & \text{if } d = (4, q-1) \in \{1, 2\}, \\ \mathcal{A}_2^4 & \text{if } d = (4, q-1) = 4. \end{cases} \tag{33}$$

with $\mathcal{A}_2^4 = \mathcal{A}_2^4(1) \cup \mathcal{A}_2^4(2)$, where

$$\mathcal{A}_2^4(1) = \left\{\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)\right\},$$
$$\mathcal{A}_2^4(2) = \left\{\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)\right\}. \tag{34}$$

Table 2
Coefficients $\beta_1(p) = N_{1,p}(E) - (p+1)$ of the Hasse–Weil modular form of the elliptic quartic curve E in terms of the cardinalities $N_{1,p}(E)$ for the lower rational primes

	Prime p												
	2	3	5	7	11	13	17	19	23	29	31	37	41
$N_{1,p}$	3	4	4	8	12	20	16	20	24	20	32	36	32
$\beta_1(p)$	0	0	−2	0	0	6	−2	0	0	−10	0	−2	−10

The sums $j_q(\alpha)$ are constant over the permutation orbits of $\mathcal{A}_2^4$, and with $j_p\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right) = \overline{j_p\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)}$ one finds

$$L_{\mathrm{HW}}(C_4, s) = \prod_p \frac{1}{(1 + 2a_p \cdot p^{-s} + p \cdot p^{-2s})^3}, \tag{35}$$

where $a_p = \mathrm{Re}\ j_p(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

Hence $L_{\mathrm{HW}}(C_4, s) = L^3(s)$ in terms of an L-function

$$L(s) = \prod_p \frac{1}{1 + 2a_p \cdot p^{-s} + p \cdot p^{-2s}} \tag{36}$$

which has the form of the Hasse–Weil series of an elliptic curve. The quartic curve C_4 has genus $g = 3$, hence this factorization indicates that the L-function of this curve splits into a product of g elliptic contributions. Such a factorization into elliptic factors will not happen for general Fermat curves.

2.4 The above decomposition allows us to return to the question, asked above, whether one can determine an elliptic curve whose Hasse–Weil series has been determined by (36). The quartic curve C_4 becomes singular when reduced at $p = 2$. Hence we would expect the purported elliptic curve to have a conductor that is divisible by two, suggesting an elliptic curve of degree 4. Such a curve is the well known weighted torus, described by

$$E_4 = \{(x_0 : x_1 : x_2) \in \mathbb{P}_{(1,1,2)} | x_0^4 + x_1^4 + x_2^2 = 0\}. \tag{37}$$

The cardinalities for this curve are collected in Table 2.

The Hasse–Weil modular form resulting from the solutions $E/\mathbb{F}_p$ starts out as

$$f_{\mathrm{HW}}(E_4, q) = q + 2q^5 - 3q^9 - 6q^{13} + 2q^{17} - q^{25} + 10q^{29} + 2q^{37} + 10q^{41} + \cdots \tag{38}$$

in agreement with the expansion $f(q)$ associated to the ‘cube root’ $L(s)$ discussed above.

We are therefore in the same situation as in [5] and we can ask whether the modular form $f(q) = f_{\mathrm{HW}}(E_4, q)$ can be related to modular forms that are induced by conformal field theoretic characters.

3. Affine Kac–Moody algebras

3.1 A construction of non-twisted affine Kac–Moody algebras, also called affine Lie algebras, is provided by the extension [12]

$$\hat{\underline{\mathrm{G}}} = \mathrm{L}\underline{\mathrm{G}} \oplus \mathbb{C}k \oplus \mathbb{C}d \tag{39}$$

of the loop algebra

$$\mathrm{L}\underline{\mathrm{G}} = \underline{\mathrm{G}} \otimes \mathbb{C}[t, t^{-1}] \tag{40}$$

by the central extension K and $D = t\frac{\mathrm{d}}{\mathrm{d}t}$. In terms of the generators $J^a \otimes t^m$ the algebra becomes

$$[J^a \otimes t^m, J^b \otimes t^n] = if_c^{ab} J^c \otimes t^{m+n} + Km\delta^{ab}\delta_{m+n,0}. \tag{41}$$

The representations of this algebra can be parametrized by affine weights $\hat{\lambda} = (\lambda, k, n)$ of the Cartan subalgebra $\{H_0^i, K, D\}$, with $i = 1, \ldots, r$, where r denotes the rank of the underlying Lie algebra $\underline{\mathrm{G}}$. For fixed level k of the theory the characters $\chi_{\hat{\lambda}}$ are essentially parametrized by the weight λ of the representation. For the reduced characters of an affine Lie algebra $\hat{\underline{\mathrm{G}}}$ at level k the characters transform as

$$\chi_{\hat{\lambda}}(-1/\tau) = \sum_{\hat{\mu} \in P_+^k} S_{\hat{\lambda},\hat{\mu}} \chi_{\hat{\mu}}(\tau), \tag{42}$$

where the modular S-matrix takes the form

$$S_{\hat{\lambda},\hat{\mu}} = \frac{i^{|\Delta_+|}}{\sqrt{|P/Q^\vee|(k+g)^r}} \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{-2\pi i(\langle w(\lambda+\rho),\mu+\rho\rangle)/(k+g)}. \tag{43}$$

Here $P = \sum_i \mathbb{Z}\omega_i$ denotes the lattice of fundamental weights ω_i defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ via coroots $\alpha_j^\vee = \frac{2\alpha}{\langle \alpha,\alpha \rangle}$. $Q^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$ is the coroot lattice and $P/Q^\vee$ denotes the lattice points of P lying in an elementary cell of $Q^\vee$, while $|P/Q^\vee|$ describes the number of points in this set. P_+^k is the set of all dominant weights at level k, and $\epsilon(w) = (-1)^{\ell(w)}$ is the signature of the Weyl group element w, where $\ell(w)$ is the minimum number of simple Weyl reflections that w decomposes into. Δ_+ is the number of positive roots in the Lie algebra $\underline{\mathrm{G}}$.

3.2 Supersymmetric string models can be constructed in terms of conformal field theories with $N = 2$ supersymmetry. The simplest class of $N = 2$ supersymmetric exactly solvable theories is built in terms of the affine $\mathrm{SU}(2)_k$ algebra at level k as a coset model $\mathrm{SU}(2)_k \otimes \mathrm{U}(1)_2/\mathrm{U}(1)_{k+2,\mathrm{diag}}$. Coset theories G/H lead to central charges of the form $c_G - c_H$, hence the supersymmetric affine theory at level k still has central charge $c_k = 3k/(k+2)$. The spectrum of anomalous dimensions $\Delta_{q,s}^\ell$ and U(1)-charges Q^ℓ of the primary fields $\Phi_{q,s}^\ell$ at level k is given by

$$\Delta_{q,s}^\ell = \frac{\ell(\ell+2) - q^2}{4(k+2)} + \frac{s^2}{8},$$

$$Q^{\ell}_{q,s} = -\frac{q}{k+2} + \frac{s}{2}, \tag{44}$$

where $\ell \in \{0, 1, \ldots, k\}$, $\ell + q + s \in 2\mathbb{Z}$, and $|q - s| \leq \ell$. Associated to the primary fields are characters defined as

$$\chi^{k}_{\ell,q,s}(\tau, z, u) = \mathrm{e}^{-2\pi i u}\mathrm{tr}_{\mathcal{H}^{\ell}_{q,s}}\, \mathrm{e}^{2\pi i \tau(L_0 - c/24)}\mathrm{e}^{2\pi i J_0}, \tag{45}$$

where the trace is to be taken over a projection $\mathcal{H}^{\ell}_{q,s}$ to a definite fermion number (mod 2) of a highest weight representation of the (right-moving) $N = 2$ algebra with highest weight vector determined by the primary field. It is of advantage to express these maps in terms of the string functions and theta functions, leading to the form

$$\chi^{k}_{\ell,q,s}(\tau, z, u) = \sum c^{k}_{\ell,q+4j-s}(\tau)\theta_{2q+(4j-s)(k+2),2k(k+2)}(\tau, z, u). \tag{46}$$

It follows from this representation that the modular behavior of the $N = 2$ characters decomposes into a product of the affine SU(2) structure in the ℓ index and into Θ-function behavior in the charge and sector index. The string functions $c^{k}_{\ell,m}(\tau)$ are given by

$$c^{k}_{\ell,m}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{\substack{-|x|<y\leq|x| \\ (x,y)\ \mathrm{or}\ \left(\frac{1}{2}-x,\frac{1}{2}+y\right) \\ \in\mathbb{Z}^2+\left(\frac{\ell+1}{2(k+2)},\frac{m}{2k}\right)}} \mathrm{sign}(x)\, \mathrm{e}^{2\pi i\tau((k+2)x^2-ky^2)} \tag{47}$$

while the classical theta functions $\theta_{m,k}$ are defined as

$$\theta_{n,m}(\tau, z, u) = \mathrm{e}^{-2\pi i m u} \sum_{\ell\in\mathbb{Z}+\frac{n}{2m}} \mathrm{e}^{2\pi i m\ell^2\tau+2\pi i\ell z}. \tag{48}$$

It follows from the coset construction that the essential ingredient in the conformal field theory is the SU(2) affine theory.

3.3 It was suggested by Gepner some time ago that exactly solvable string compactifications obtained by tensoring several copies of $N = 2$ supersymmetric conformal field theories derived from affine Kac–Moody algebras should yield, after performing appropriate projections, theories that correspond in some limit to geometric compactification described by Brieskorn–Pham type Calabi–Yau varieties [13]. The evidence for this conjecture was initially based mostly on spectral information for all models in the Gepner class of solvable string compactifications and the agreement of certain types of intersection numbers which allow an interpretation as Yukawa couplings, as well as Landau–Ginzburg type arguments [14]. In the case of the Fermat cubic curve these results suggest that there is an underlying conformal field theory of this elliptic curve that is described by the GSO projection of a tensor product of three models at conformal level $k = 1$. Roughly, this entails a relation of the type

$$\mathbb{P}_2 \supset C_3 \leftrightarrow (\mathrm{SU}(2)_{k=1,A_1})^{\otimes 3}_{\mathrm{GSO}}, \tag{49}$$

where A_1 signifies the diagonal invariant for the SU(2) partition function, and GSO indicates the projection which guarantees integral U(1)-charges of the states.

In the case of higher genus curves no such direct relation is expected from a string theory perspective. There is, however, a weaker embedding argument that suggests a possible modular interpretation. Higher genus curves can be embedded in higher dimensional Calabi–Yau varieties which in turn are conjectured to be exactly solvable. Examples of such embeddings for the quartic curve are provided by threefolds such as Brieskorn–Pham hypersurfaces $\mathbb{P}_{(1,1,2,2,2)}$ [8], $\mathbb{P}_{(1,2,3,3,3)}$ [12], $\mathbb{P}_{(1,4,5,5,5)}$ [20], and others. The first of these varieties, e.g.

$$X_8 = \{z_0^8 + z_1^8 + z_2^4 + z_3^4 + z_4^4 = 0\} \subset \mathbb{P}_{(1,1,2,2,2)} \tag{50}$$

is expected to be related to the conformal field theory

$$((\mathrm{SU}(2)_{k=6,A_1})^{\otimes 2} \otimes (\mathrm{SU}(2)_{k=2,A_1})^{\otimes 3})_{\mathrm{GSO}}. \tag{51}$$

Hence we might expect that conformal field theoretic aspects are encoded in the quartic curve

$$\mathbb{P}_2 \supset C_4 \leftrightarrow (\mathrm{SU}(2)_{k=2,A_1})^{\otimes 3}. \tag{52}$$

This leads to the affine Kac–Moody algebra SU(2) at level two and we can ask whether there are relations between the modular objects determined by the affine algebra, and the modular objects determined by the variety. The results of [5] suggest that interesting affine quantities to consider are the SU(2) theta functions associated to string functions at conformal level k, defined as $\Theta^k_{\ell,m}(\tau) = \eta^3(\tau)c^k_{\ell,m}(\tau)$. The expansion of the theta functions at conformal level $k = 2$ follows from those of the string function expansions. It turns out that relevant for the present discussion is the theta function

$$\Theta^2_{1,1}(q) = q^{1/8}(1 - q - 2q^2 + q^3 + 2q^5 + \mathcal{O}(q^6)). \tag{53}$$

This is a modular form of weight one, and therefore cannot be identified directly with the Hasse–Weil form itself. It will become clear below, however, that $\Theta^2_{1,1}(q)$ emerges as the building block of the Hasse–Weil modular form $f_{\mathrm{HW}}(C_4, q)$ of the quartic plane Fermat curve.

4. Geometric modularity

4.1 From a physical perspective it is not clear a priori which conformal field theoretic quantities should be the correct building blocks of the Hasse–Weil function, if any. Possibilities include the twists of the affine or parafermionic characters, or some elements of the $N = 2$ superconformal model. The coset construction shows that the most important ingredient of the $N = 2$ theory is given by the affine SU(2) Kac–Moody algebra. The string functions $c^k_{\ell,m}(\tau)$ of the $N = 2$ characters would appear to be natural candidates because they capture the essential interacting nature of the field theory. Furthermore, associated to string functions are natural number theoretic theta functions.

The main result of [5] shows that the Hasse–Weil L-function of the cubic plane curve C_3 is determined by the Kac–Moody string function of the affine SU(2) algebra at conformal level $k = 1$. At $k = 1$ there is only one string function, $c^1_{1,1}(\tau)$, which can be computed to lead to the expansion

$$c^1_{1,1}(\tau) = q^{-1/24}(1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \mathcal{O}(q^6)). \tag{54}$$

It turns out that more important than the string functions are the associated SU(2) theta functions $\Theta^k_{\ell,m}(\tau)$ (3). These objects are associated to quadratic number fields determined by the level of the affine theory [15]. At level $k = 1$ the unique theta function $\Theta^1_{1,1}(\tau)$ is associated to the real quadratic extension $\mathbb{Q}(\sqrt{3})$ of the rational field $\mathbb{Q}$. Its expansion follows from the string function expansion, resulting in

$$\Theta^1_{1,1}(q) = q^{1/12}(1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 + \mathcal{O}(q^6)). \tag{55}$$

It is this modular form of weight one which emerged in [5] as the building block of the Hasse–Weil modular form $f_{\mathrm{HW}}(C_3, q)$ of the cubic elliptic curve

$$f_{\mathrm{HW}}(C_3, \tau) = \Theta^1_{1,1}(3\tau)\Theta^1_{1,1}(9\tau). \tag{56}$$

4.2 The case of the elliptic cubic curve is special because it is a Calabi–Yau curve and therefore defines a consistent string theory background. For higher genus curves this is not the case, and it is therefore not a priori clear whether one should expect a relation to the string theoretic Kac–Moody algebra. As mentioned above, an argument that is encouraging is that higher genus algebraic curves such as the quartic, and others, appear in higher dimensional Calabi–Yau varieties as singular curves which have to be resolved.

4.3 In the case of the quartic curve we can use the result derived above that the L-function is that of a triple product of elliptic curves. This reduces the current problem to the type of problem solved in [5], and we can follow the logic used in the construction introduced there. First we need to determine the weight and the level of this form. For a general elliptic curve the corresponding modular form $f_{\mathrm{HW}}(E, q)$ defined by the Mellin transform of the Hasse–Weil L-function $L_{\mathrm{HW}}(E, s)$ is determined by the proof of the Shimura–Taniyama conjecture to be an element in $S_2(\Gamma_0(N))$ for some level N. Alternatively, one can read off the weight of a Hecke eigenform from the multiplicative properties of its coefficients $\{a_n\}_{n\in\mathbb{N}}$ induced by the Hecke operators

$$\begin{aligned} &a_{mn} = a_m a_n (m,n) = 1, \qquad a_{p^{n+1}} = a_{p^n} a_p - p^{k-1} a_{p^{n-1}}, \\ &a_{p^n} = (a_p)^n, \quad \text{for } p|N, \end{aligned} \tag{57}$$

as described in more detail in [5] for the case of cubic plane curve.

4.4 The quartic curve is singular at the prime $p = 2$, therefore we expect the conductor of the elliptic curves involved to be divisible by 2, and perhaps by powers of two. Using Weil's conductor conjecture [16] for elliptic curves we further expect the modular conductor of the corresponding modular form to be divisible by some power of 2. The arithmetic conductor of the elliptic curve E can be determined via Tate's algorithm [17] from the generalized affine form of E given by

$$y^2 - a_1 xy - a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \tag{58}$$

By a result of Ogg [18] the conductor of such a curve can be computed as

$$N_{E/\mathbb{Q}} = \prod_{\text{bad } p} p^{f_p}, \tag{59}$$

where the exponent f_p is given by

$$f_p = \mathrm{ord}_p \Delta_{E/\mathbb{Q}} + 1 - s_p, \tag{60}$$

in terms of discriminant $\Delta_{E/\mathbb{Q}}$ of the curve and the number s_p of irreducible components of the singular fiber at p. This shows that the conductor of the elliptic curve can be viewed as a quantity which combines the bad primes with a measure of the severity of the singularity at these bad primes.

4.5 The elliptic quartic curve E_4 can be transformed into its affine form by choosing inhomogeneous coordinates and defining new coordinates $v = xz/y$ and $u = (x/y)^2$, leading to the Weierstrass form

$$v^2 = u^3 + u. \tag{61}$$

Tate's computation of the discriminant simplifies considerably for this curve, leading to $\Delta_{E_4} = -64$, and therefore $\mathrm{ord}_2 \Delta_{E_4} = 6$. The singular fiber is of Kodaira type II, and therefore the arithmetic conductor is $N = 64$.

4.6 Combining the conductor argument with the string theoretic embedding of the quartic curve into higher dimensional varieties we are led to consider the form

$$\Theta_{1,1}^2(q^4)^2 = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} - 10q^{29} + \cdots \tag{62}$$

Comparing this with the Mellin transform (38) of the cubic root of the Hasse–Weil L-form of the quartic C_4 shows agreement with $\Theta_{1,1}^2(4\tau)^2$ except for the signs in the terms q^n with $n = 1 \,(\mathrm{mod}\, 8)$.

We therefore see that the discussion of the quartic curve must involve an ingredient that goes beyond the analysis that succeeded in [5] in providing a string theoretic Kac–Moody algebra interpretation of the Hasse–Weil L-function. In some sense the elliptic curve defined by C_3 is too special to reveal all the key elements necessary for this identification. In the following we will first complete the identification of the Hasse–Weil modular form with a string theoretic modular form in a somewhat utilitarian manner by pin-pointing the missing ingredient. We then turn to the physical interpretation of this ingredient and explain why it does not appear in the discussion of C_3.

The sign flip suggests that the Hasse–Weil modular form $f_{\mathrm{HW}}(E_4, q)$ might be related to the modular form $\Theta_{1,1}^2(4\tau)^2$ via a twist. For a modular form $f(q) = \sum_n a_n q^n$ and a Dirichlet character

$$\chi : \mathbb{Z} \to K^{\times} \tag{63}$$

with values in a field K, define the twisted form as

$$f_\chi(q) = \sum_n \chi(n) a_n q^n. \tag{64}$$

We are interested in a character with conductor 8, taking values in $K = \mathbb{F}_2$. An example of a class of characters which leads to such an object is provided by Legendre symbols. These are defined on rational primes as

$$\chi_n(p) = \left(\frac{n}{p}\right) = \begin{cases} 1 & n \text{ is a square in } \mathbb{F}_p, \\ -1 & n \text{ is not a square in } \mathbb{F}_p. \end{cases} \tag{65}$$

The conductor of $\chi_n(\cdot)$ is given by n if $n = 1\,(\mathrm{mod}\,4)$ and $4n$ for $n = 2, 3\,(\mathrm{mod}\ 4)$.

For non-prime numbers the generalized Legendre symbol is defined by using the prime decomposition. Every natural number m can be decomposed into primes as $m = p_1 \cdots p_r$ and the generalized symbol is defined as

$$\chi_n(m) = \prod_{i=1}^{r} \left(\frac{n}{p_i}\right). \tag{66}$$

This shows that for $n = 2$ the Legendre symbol takes the form

$$\chi_2(r) = \left(\frac{2}{r}\right) = \begin{cases} 1 & r = 1\,(\mathrm{mod}\,8), \\ -1 & r = 5\,(\mathrm{mod}\,8). \end{cases} \tag{67}$$

It is this character which provides the twist from the conformal field theory induced modular form to the elliptic L-function which provides the basic building block of the Hasse–Weil L-function of the quartic curve.

We therefore see that the CFT modular form $\Theta^2_{1,1}(4\tau)^2$ of weight two is the twist of the geometric weight two modular form via the number theoretic Legendre symbol. Put differently, we see that if we denote by $\Theta^2_{1,1}(4\tau)^2 \otimes \chi_2$ the modular form obtained by twisting the form $\Theta^2_{1,1}(4\tau)^2$ by the character $\chi_2(\cdot)$, we can write the Hasse–Weil L-function of as

$$L_{\mathrm{HW}}(C_4, s) = L(\Theta^2_{1,1}(4\tau)^2 \otimes \chi_2, s)^3. \tag{68}$$

This concludes the proof of the theorem stated in Section 1.

The remaining question is whether the theta function input is unique. To answer this one can use the Eichler–Shimura theory [19–21]. $\Theta^2_{1,1}(4\tau)^2$ is an element in the space of cusp forms $S_2(\Gamma_0(32))$. For arbitrary modular level N, the dimension of the space $S_2(\Gamma_0(N))$ can be computed as the genus of the modular curve $X_0(N)$ via

$$g(X_0(N)) = 1 + \frac{\mu(N)}{12} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} - \frac{\nu_\infty}{2}, \tag{69}$$

where $\mu(N)$ is the index of $\Gamma_0(N)$ in $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$, $\nu_2(N)$ and $\nu_3(N)$ are the number of elliptic points of order 2 and 3, and $\nu_\infty(N)$ is the number of $\Gamma_0(N)$ inequivalent cusps. For $N = 32$ this implies that the space of cusp forms is one-dimensional, hence $\Theta^1_1(4\tau)^2$ is its unique generator, up to the multiplication of a constant.

5. Quantum dimensions

The character $\chi_2(\cdot)$ which appears above in the context of providing a string theoretic interpretation of the Hasse–Weil modular form furthermore points toward the field of quantum dimensions, thereby providing a link between the problem of a geometric explanation of the string theoretic modularity and the problem of providing a geometric explanation of the string theoretic spectrum.

For general square free n the Legendre characters $\chi_n(\cdot)$ allow a characterization of the factorization behavior of rational primes p when viewed as elements in a quadratic extension $\mathbb{Q}(\sqrt{n})$. They describe whether the rational prime p splits into a product of prime ideals $\mathfrak{p}_i \subset \mathcal{O}_{\mathbb{Q}(\sqrt{n})}$ in the ring of algebraic integers $\mathcal{O}_{\mathbb{Q}(\sqrt{n})}$. The principal ideal (p) factors as $(p) = \mathfrak{p}_1\mathfrak{p}_2$ if $\chi_n(p) = 1$, remains prime if $\chi_n(p) = -1$, and ramifies, i.e. $(p) = \mathfrak{p}^2$, if $\chi_n(p) = 0$, i.e. $p|n$. In summary we can write

$$\chi_n(p) = \begin{cases} 1 & \text{if } (p) = \mathfrak{p}_1\mathfrak{p}_2, \\ -1 & \text{if } (p) \text{ is prime}, \\ 0 & \text{if } (p) = \mathfrak{p}^2. \end{cases} \tag{70}$$

For $n = 2$ the character χ_2 therefore is associated to the quadratic extension $\mathbb{Q}(\sqrt{2})$. It turns out that this is precisely the field determined by the anomalous dimensions of the affine theory when mapped into the quantum dimensions. This can be seen as follows.

It was noted in [1] that a link between the geometry and the conformal field theory can be obtained by using a translation of the anomalous dimensions into algebraic integers via the Rogers dilogarithm. In the simple case of the affine Lie algebra SU(2) at conformal level k the modular transformations of the characters χ associated to the primary fields Φ_i with anomalous dimensions

$$\Delta_\ell = \frac{\ell(\ell+2)}{4(k+2)} \tag{71}$$

are given by

$$\chi_\ell\left(-\frac{1}{\tau}, \frac{u}{\tau}\right) = \mathrm{e}^{\pi i k u^2/2} \sum_m S_{\ell m}\chi_m(\tau, u) \tag{72}$$

with modular S-matrix

$$S_{\ell m} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(\ell+1)(m+1)\pi}{k+2}\right), \quad 0 \le \ell, m \le k. \tag{73}$$

With these matrices one can define the generalized quantum dimensions as $Q_{\ell m} = S_{\ell m}/S_{0m}$ for the affine SU(2) algebra at level k. The importance of these numbers derives from the fact that even though they do not directly provide the scaling behavior of the correlation functions, they do contain the complete information about the anomalous dimensions as well as the central charge. The first step in this direction was the realization by Kirillov and Reshetikhin that the central charge can be expressed in terms of the quantum dimensions. Earlier mathematical results had been obtained by Lewin. Denote by L Rogers' dilogarithm

$$L(z) = Li_2(z) + \tfrac{1}{2}\log(z)\,\log(1-z) \tag{74}$$

and Li_2 is Euler's classical dilogarithm

$$Li_2(z) = \sum_{n\in\mathbb{N}} \frac{z^n}{n^2}. \tag{75}$$

Then one has the following result.

Theorem ([22–24]). *For the generalized quantum dimensions $Q_{\ell m}$ one finds the following relations*

$$\frac{1}{L(1)}\sum_{\ell=1}^{k} L\left(\frac{1}{Q_{\ell m}^2}\right) = \frac{3k}{k+2} - 24\Delta_m^{(k)} + 6m. \tag{76}$$

For $m = 0$ this theorem reduces to the central charge result in terms of the quantum dimensions $Q_\ell = S_{\ell 0}/S_{00}$

$$\frac{1}{L(1)}\sum_{\ell=1}^{k} L\left(\frac{1}{Q_{\ell}^2}\right) = \frac{3k}{k+2}, \tag{77}$$

that was obtained earlier in [25,26].

It follows that the quantum dimensions contain the essential information about the spectrum of the conformal field theory and Rogers' dilogarithm provides, via Euler's dilogarithm, the map from the quantum dimensions to the central charge and the anomalous dimensions. A review of these results and references to the original literature can be found in [24].

In the case of the elliptic curve E_4 the exactly solvable model is a tensor product of two SU(2) theories at conformal level $k = 2$, equipped with the diagonal affine invariant

$$\mathbb{P}_{(1,1,2)} \supset E_4 \sim (\mathrm{SU}(2)^{\otimes 2}_{k=2,A})_{\mathrm{GSO}}. \tag{78}$$

The quantum dimensions of the SU(2) theory at level $k = 2$ take values in the quadratic extension $\mathbb{Q}(\sqrt{2})$ and therefore we find that the field of quantum dimensions provides a physical explanation of the emergence of the Legendre character in the modularity relation (13). Hence the main result of this paper can be interpreted as a geometric derivation of a conformal field theoretic object-the conformal field theoretic modular form $\Theta^2_{1,1}(4\tau)^2$ is the twist of the geometric modular form defined by the Hasse–Weil L-function via the quadratic character associated to the number field of the quantum dimensions.

This physical explanation of the origin of the twist makes it clear why no character was necessary in the discussion in [5] of the plane cubic elliptic curve C_3. The quantum dimensions of the affine theory $A_1^{(1)}$ at conformal level $k = 1$ take values in the field of rational number $\mathbb{Q}$.

6. Generalizations

For more general curves of higher genus it is possible to decompose the Jacobians in a way similar to the analysis in previous sections. In general the factorization behavior can be quite involved and in the following we briefly describe what is known about the splitting behavior. For more detail we refer to the original literature [27,30] (see also [31,2]). The reason why this factorization behavior is relevant in the present context is that the splitting occurs on the level of isogenies, a map between abelian varieties that is weaker than an isomorphism. It is however known that isogeneous varieties have the same L-functions. Also important is that the varieties that emerge in this factorization admit complex multiplication and therefore their L-functions are determined by algebraic Hecke characters. Hecke L-functions are known to be modular and therefore the factorization of Jacobian varieties allows to determine modular forms associated to these higher genus curves.

It was shown by Faddeev [27][1] that the Jacobian variety $J(C_d)$ of Fermat curves $C_d \subset \mathbb{P}_2$ of prime degree splits into a product of abelian factors $A_{\mathcal{O}_i}$

$$J(C_d) \cong \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i}, \tag{79}$$

where the set $\mathcal{I}$ provides a parametrization of the cohomology of C_d, and the sets $\mathcal{O}_i$ are orbits in $\mathcal{I}$ of the multiplicative subgroup $(\mathbb{Z}/d\mathbb{Z})^\times$ of the group $\mathbb{Z}/d\mathbb{Z}$. More precisely it was shown that there is an isogeny

$$i : J(C_d) \to \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i}, \tag{80}$$

where an isogeny $i : A \to B$ between abelian varieties is defined to be a surjective homomorphism with finite kernel. Explicitly, $\mathcal{I}$ is the set of triplets (r, s, t) parametrizing a basis of the cohomology

$$\mathrm{H}^1(C_d) = \{\omega_{r,s,t} = x^{r-1} y^{s-d}\, \mathrm{d}x \,|\, r, s, t \in \mathbb{N}, 0 < r, s, t < d, r + s + t = 0 \bmod d\}, \tag{81}$$

and the abelian varieties $A_d^{[(r,s,t)]}$ are associated to orbits $[(r, s, t)]$ of the triplets (r, s, t) with respect to the group $(\mathbb{Z}/d\mathbb{Z})^\times$.

The periods of the Fermat curve have been computed by Rohrlich [30] to be

$$\int_{\mathcal{A}^j \mathcal{B}^k \kappa} \omega_{r,s,t} = \frac{1}{d} B\left(\frac{s}{d}, \frac{t}{d}\right) (1 - \xi^s)(1 - \xi^t)\xi^{js+kt}, \tag{82}$$

where ξ is a primitive dth root of unity, and

$$B(u, v) = \int_0^1 t^{u-1}(1 - v)^{v-1}\, \mathrm{d}t \tag{83}$$

is the classical beta function. $\mathcal{A}$, $\mathcal{B}$ are the two automorphism generators

$$\mathcal{A}(1, y, z) = (1, \xi y, z), \qquad \mathcal{B}(1, y, z) = (1, y, \xi z) \tag{84}$$

and κ is the generator of $\mathrm{H}_1(C_d)$ as a cyclic module over $\mathbb{Z}[\mathcal{A}, \mathcal{B}]$. The period lattice of the Fermat curve therefore is the span of

$$\left(\ldots, \xi^{jr+ks}(1 - \xi^r)(1 - \xi^s)\frac{1}{d} B\left(\frac{r}{d}, \frac{s}{d}\right), \ldots\right)_{\substack{1 \le r,s,t \le d-1 \\ r+s+t=d}}, \quad \forall 0 \le j, k \le d - 1. \tag{85}$$

[1] More accessible references are [28–30].

The abelian factor $A_{[(r,s,t)]}$ associated to the orbit $\mathcal{O}_{r,s,t} = [(r, s, t)]$ can be obtained as the quotient

$$A_{[(r,s,t)]} = \mathbb{C}^{\varphi(d_0)/2}/\Lambda_{r,s,t}, \tag{86}$$

where $d_0 = d/\gcd(r, s, t)$ and the lattice $\Lambda_{r,s,t}$ is generated by elements of the form

$$\sigma_a(z)(1-\xi^{as})(1-\xi^{at})\frac{1}{d}B\left(\frac{\langle as\rangle}{d}, \frac{\langle at\rangle}{d}\right), \tag{87}$$

where $z \in \mathbb{Z}[\mu_{d_0}]$, $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\mu_{d_0})/\mathbb{Q})$ runs through subgroups of the Galois group of the cyclotomic field $\mathbb{Q}(\mu_{d_0})$ and $\langle x\rangle$ is the smallest integer $0 \leq x < 1$ congruent to x mod d.

Alternatively, the abelian variety $A_d^{r,s,t}$ can be constructed in a more geometric way as follows. Consider the orbifold of the Fermat curve C_d with respect to the group defined as

$$G_d^{r,s,t} = \{(\xi_1, \xi_2, \xi_3) \in \mu_d^3 | \xi_1^r \xi_2^s \xi_3^t = 1\}. \tag{88}$$

The quotient $C_d/G_d^{r,s,t}$ can be described algebraically via projections

$$T_d^{r,s,t} : C_d \to C_d^{r,s,t}, \qquad (x, y) \mapsto (x^d, x^r y^s) =: (u, v), \tag{89}$$

which map C_d into the curves

$$C_d^{r,s,t} = \{v^d = u^r(1-u)^s\}. \tag{90}$$

For prime degrees the abelian varieties $A_d^{r,s,t}$ can be defined simply as the Jacobians $J(C_d^{r,s,t})$ of the projections $C_d^{r,s,t}$. When d has non-trivial divisors $m|d$, this definition must be modified as follows. Consider the projected Fermat curves

$$C_d \to C_m, \qquad (x, y) \mapsto (\overline{x}, \overline{y}) := (x^{d/m}, y^{d/m}), \tag{91}$$

whose Jacobians can be embedded as $e : J(C_m) \to J(C_d)$. Composing the projection $T_d^{r,s,t}$ as

$$J(C_m) \stackrel{e}{\to} J(C_d) \stackrel{T_d^{r,s,t}}{\to} J(C_d^{r,s,t}) \tag{92}$$

for all proper divisors $m|d$ leads to a collection of subvarieties $\cup_{m|d} T_d^{r,s,t}(e(J(C_m)))$. The abelian variety of interest then is defined as

$$A_d^{r,s,t} = J(C_d^{r,s,t})/\cup_{m|d} T_d^{r,s,t}(e(J(C_m))). \tag{93}$$

The abelian varieties $A_d^{r,s,t}$ are not necessarily simple but it can happen that they in turn can be factored. This question can be analyzed via a criterion of Shimura–Taniyama, described in [32]. Applied to the $A_d^{r,s,t}$ discussed here the Shimura–Taniyama criterion involves computing for each set $H_d^{r,s,t}$ defined as

$$\mathrm{H}_d^{r,s,t} := \{a \in (\mathbb{Z}/d\mathbb{Z})^\times | \langle ar\rangle + \langle as\rangle + \langle at\rangle = d\ \} \tag{94}$$

another set $W_d^{r,s,t}$ defined as

$$W_d^{r,s,t} = \{a \in (\mathbb{Z}/d\mathbb{Z})^\times | aH_d^{r,s,t} = H_d^{r,s,t}\}. \tag{95}$$

If the order $|W_d^{r,s,t}|$ of $W_d^{r,s,t}$ is unity then the abelian variety $A_d^{r,s,t}$ is simple, otherwise it splits into $|W_d^{r,s,t}|$ factors [33].

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References

[1] R. Schimmrigk, Arithmetic of Calabi–Yau varieties and rational conformal field theory, J. Geom. Phys. 44 (2003) 555–565 [arXiv: hep-th/0111226].
[2] M. Lynker, R. Schimmrigk, S. Stewart, Complex multiplication of exactly solvable Calabi–Yau varieties, Nucl. Phys. B700 (2004) 463–489 [hep-th/0312319].
[3] A. Wiles, Modular elliptic curves and Fermat's last theorem, Ann. Math. 141 (1995) 443–551;
R. Taylor, A. Wiles, Ring-theoretic properties of certain Hecke algebras, Ann. Math. 141 (1995) 553–572.
[4] C. Breuil, B. Conrad, F. Diamond, R. Taylor, On the modularity of elliptic curves over $\mathbb{Q}$, or wild 3-adic exercises, J. Am. Math. Soc. 14 (2001) 843–939.
[5] R. Schimmrigk, S. Underwood, The Shimura–Taniyama conjecture and conformal field theory, J. Geom. Phys. 48 (2003) 169–189 [arXiv: hep-th/0211284].
[6] B. Hunt, R. Schimmrigk, K3-fibered Calabi–Yau threefolds. I. The twist map, Int. J. Math. 10 (1999) 833–869 [arXiv: math.AG/9904059].
[7] P. Candelas, M. Lynker, R. Schimmrigk, Calabi–Yau manifolds in weighted $\mathbb{P}_4$, Nucl. Phys. B 341 (1990) 383–402.
[8] A. Klemm, R. Schimmrigk, Landau–Ginzburg string vacua, Nucl. Phys. B 411 (1994) 559–583 [arXiv: hep-th/9204060];
M. Kreuzer, H. Skarke, No mirror symmetry in Landau–Ginzburg spectra! Nucl. Phys. B 388 (1992) 113–130 [arXiv: hep-th/9205004].
[9] F.K. Schmidt, Analytische Zahlentheorie in Körpern der Charakteristik p, Math. Z. 33 (1931) 1–32.
[10] H. Hasse, Über die Kongruenzzetafunktionen. Unter Benutzung von Mitteilungen von Prof. Dr. F.K. Schmidt und Prof. Dr. E. Artin, S. Ber. Preuß. Akad. Wiss. H. 17 (1934) 250–263.
[11] A. Weil, Jacobi sums as "Grössencharaktere", Trans. Am. Math. Soc. 73 (1952) 487.
[12] V.G. Kac, Infinite Lie Algebras, Cambridge University Press, 1990.
[13] D. Gepner, Spacetime supersymmetry in compactified string theory and superconformal models, Nucl. Phys. B (1988) 757.
[14] E. Martinec, Algebraic geometry and effective Lagrangians, Phys. Lett. B 217 (1989) 431;
C. Vafa, N. Warner, Catastrophes and the classification of conformal field theories, Phys. Lett. B 218 (1989) 51;
W. Lerche, C. Vafa, N. Warner, Chiral rings in $N = 2$ superconformal theories, Nucl. Phys. B 324 (1989) 427.
[15] V.G. Kac, D.H. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, Adv. Math. 53 (1984) 125–264.
[16] A. Weil, Uber die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. 168 (1967) 149–156.
[17] J. Tate, Algorithm for determining the type of a singular fiber in on elliptic pencil, Letter to J. Cassels, in: B.J. Birch, W. Kuyk (Eds.), Modular Functions of One Variable IV, Springer Verlag, LNM 476, 1975.
[18] A. Ogg, Elliptic curves and wild ramifications, Am. J. Math. 89 (1967) 1–21.

[19] M. Eichler, Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion, Arch. Math. 5 (1954) 355–366.
[20] G. Shimura, Correspondences modulaires et les fonction ζ de courbes algebrique, J. Math. Soc. Jpn. 10 (1958) 1–28.
[21] G. Shimura, Sur les intégrales attachés aux formes automorphes, J. Math. Soc. Jpn. 11 (1959) 291–311.
[22] W. Nahm, A. Recknagel, M. Terhoeven, Dilogarithm identities in conformal field theory, Mod. Phys. Lett. A 8 (1993) 1835–1848 [arXiv: hep-th/9211034].
[23] A.N. Kirillov, Dilogarithm identities, partitions, and conformal field theory [arXiv: hep-th/9212150].
[24] A.N. Kirillov, Dilogarithm identities, Prog. Theor. Phys. Suppl. 118 (1995) 61 [arXiv: hep-th/9408113].
[25] A.N. Kirillov, N.Yu. Reshetikhin, Exact solution of the integrable XXZ Heisenberg model with arbitrary spin: I, II, J. Phys. A 20 (1987) 1565–1586, 1587–1597.
[26] L. Lewin, Polylogarithms and associated functions, 1981.
[27] D.K. Faddeev, On the divisor groups of some algebraic curves, Dokl. Tom 136 (1961) 296–298 [Sov. Math. 2 (1961) 67–69]; D.K. Faddeev, Invariants of divisor classes for the curves $y^\ell = x^k(1-x)$ in an ℓ-adic cyclotomic field, Trudy Mat. Inst. Steklova 64 (1961) 284–293 (in Russian).
[28] A. Weil, Sur les Périodes des Intégrales Abeliennes, Commun. Pure Appl. Math. 29 (1976) 813.
[29] B.H. Gross, On the periods of abelian integrals and a formula of Chowla and Selberg, Inv. Math. 45 (1978) 193.
[30] D.E. Rohrlich, The periods of the fermat curve, Appendix to Gross 1978, Inv. Math. 45 (1978) 208.
[31] M. Lynker, V. Periwal, R. Schimmrigk, Complex multiplication and black hole attractor varieties, Nucl. Phys. B 667 (2003) 484–504 [hep-th/0303111].
[32] G. Shimura, Y. Taniyama, Complex multiplication of abelian varieties and its application to number theory, 1961.
[33] N. Koblitz, D.E. Rohrlich, Simple factors in the Jacobian of a fermat curve, Can. J. Math. 30 (1978) 1183–1205.